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Asymptotic summation of Hermite series

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Abstract. A new method for the numerical evaluation of slowly convergent or even divergent series involving the Hermite functions $\psi_m(x)=(2^m m!\sqrt{\pi})^{-1/2}\,e^{-x^2/2}H_m(x)$ is presented. We consider series with either of the forms $F(x)=\Sigma_{m=0}^{\infty}c_m\psi_m(x/\sqrt{2})$ and $G(x,y)=\Sigma_{m=0}^{\infty}c_m\psi_m(x/\sqrt{2})\psi_m(y/\sqrt{2})$, where c_m decays algebraically as $m\to\infty$. The first series is a Fourier–Hermite series, while the second arises in the representation of Green functions for problems whose eigenfunctions involve the Hermite functions. By use of the Poisson summation formula, we derive rapidly convergent asymptotic expansions for the remainders of these series after a sufficiently large number of terms. The series can then be evaluated as a partial sum plus an asymptotic approximation to its remainder. The asymptotic expansion for the remainder of $G(x,y)$ also reveals the nature of the possible singular behaviour of this series near $x=y$.

1. Introduction

Although the Hermite functions $\psi_m(x)=(2^m m!\sqrt{\pi})^{-1/2}\,e^{-x^2/2}H_m(x)$ constitute a complete orthornormal set, the Fourier–Hermite expansion of a given function may converge slowly, or even diverge, requiring the use of summability methods (Hardy 1956, Wimp 1981) for its evaluation (Markett 1984, Thangavelu 1989). This paper describes a new method for the numerical evaluation of functions defined by their Fourier–Hermite expansions, which is applicable when the coefficients c_m in the expansion

$$F(x)=\sum_{m=0}^{\infty}c_m\psi_m(x/\sqrt{2}) \tag{1.1}$$

decay algebraically with m as $m\to\infty$; this choice of coefficients is motivated by existing applications of Fourier–Hermite series in geophysical fluid dynamics, in which x represents latitude (Moore and Philander 1977). The basic idea of the present method is to transform the Mth remainder of (1.1),

$$R_M(x)=\sum_{m=M}^{\infty}c_m\psi_m(x/\sqrt{2}) \tag{1.2}$$

using the Poisson summation formula (Henrici 1977), and to evaluate the transformed series asymptotically for large M, yielding an explicit and rapidly convergent expansion for the remainder (1.2). An estimate is also obtained for the range of values of M in which the asymptotic expansion for $R_M(x)$ is valid. Thus, we propose to evaluate the Fourier–Hermite series as

$$F(x)\approx\sum_{m=0}^{M}c_m\psi_m(x/\sqrt{2})+\tilde{R}_M(x) \tag{1.3}$$

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where $\tilde{R}_M(x)$ is an asymptotic approximation to the remainder (1.2). The method is able to treat convergent and divergent expansions simultaneously, since all divergent series and integrals which occur in the analysis to be presented below are evaluated by Abelian summability methods (Hardy 1956).

The present analysis somewhat resembles the spirit of Nussenzveig's (1965) study of high-frequency scattering by a rigid sphere; there, Poisson's formula and residue integration are employed to improve the convergence of a partial wave expansion of the wavefunction, which then becomes amenable to asymptotic analysis. Poisson's formula also proves very effective in accelerating the convergence of crystal lattice sums (Wimp 1981).

In this paper, we also consider series of the form

$$G(x, y) = \sum_{m=0}^{\infty} c_m \psi_m(x/\sqrt{2}) \psi_m(y/\sqrt{2}) \tag{1.4}$$

which arise in the representation of Green functions and harmonically forced solutions of the equations of motion of a barotropic equatorial ocean (Vianna and Holvorcem 1991). The eigenfunctions of these equations can be written in terms of Hermite functions (Moore and Philander 1977). In this context, the series describing the wavefields due to either a point oscillating source or a forcing distributed as a plane wave are slowly convergent or even divergent. An asymptotic expansion to the Mth remainder of (1.4) is derived in the case of algebraically decaying coefficients c_m, a choice which turns out to be adequate for the above-mentioned applications (Vianna and Holvorcem 1991). The resulting asymptotic expansion can again be used both for convergent and divergent series of the form (1.4).

In the context of representation of Green functions, it is often important to know the nature of the possible singular behaviour of $G(x, y)$ as $x \to y$. A nice feature of the present method is that, besides allowing the numerical evaluation of $G(x, y)$, the asymptotic expansion for the remainder of (1.4) yields explicitly all the singular terms which contribute to $G(x, y)$ as $x \to y$. Thus, writing

$$G(x, y) = G_{\text{sing}}(x, y) + g(x, y) \tag{1.5}$$

where $G_{\text{sing}}(x, y)$ has a singularity at $x = y$ and $g(x, y)$ is bounded, it is possible, for example, to evaluate singular integrals involving the Green function (Vijayakumar and Cormack 1988, 1989). This result is particularly useful in the numerical solution of boundary value problems by boundary integral equations (Holvorcem and Vianna 1991).

The present approach to the summation of series is related to a general method of Wimp (1974) for finding asymptotic expansions for the remainder R_M of a series whose general term a_m itself admits an asymptotic expansion $\tilde{a}_m$, as $m \to \infty$. This method is based on substituting a general asymptotic expansion $\tilde{R}_M$ with undetermined coefficients in the difference equation

$$R_{M+1} - R_M = -a_M \sim -\tilde{a}_M \tag{1.6}$$

expanding $\tilde{R}_{M+1}$ in terms of functions with argument M instead of $M+1$, and finally equating the coefficients of the various functions of M appearing on both sides of (1.6) to determine the coefficients.

The results in sections 2 and 3 on the remainder of $F(x)$ could alternatively be derived by a slightly modified version of Wimp's method, instead of using Poisson's formula. However, when treating the remainder of $G(x, y)$ (section 4), it will turn out that Wimp's method does not yield the correct behaviour of the series as $x \to y$, which limits the validity of the resulting expansion to sufficiently large $|x-y|$. This difficulty does not occur in the present method, where different asymptotic expansions naturally arise depending on the magnitude of the difference $(x-y)$ and the point of truncation M; the expansions presented in this work for the remainder of $G(x, y)$ are valid for arbitrary values of (x, y).

2. Asymptotic expansion for the remainder of $F(x)$

In this section, we study the behaviour of the terms of (1.2) when M is large, and use this information to derive an asymptotic expansion for the remainder $R_M(x)$ in terms of certain 'auxiliary series', which are asymptotically evaluated in section 3.

The Hermite functions may be written in terms of Weber parabolic cylinder functions $U(a, x)$ (Abramowitz and Stegun 1965) as

$$\psi_m(x/\sqrt{2}) = (m!\sqrt{\pi})^{-1/2} U(-\tilde{m}, x) \tag{2.1}$$

where $\tilde{m} = m + 1/2$. Uniform asymptotic expansions for $U(a, x)$ as $a \to \infty$ in the complex plane are available either in terms of elementary functions or in terms of Airy functions (Olver 1959). When $a = -\tilde{m} \to -\infty$, the former expansions are valid for $|\tau| < 1$, where

$$\tau = x/2\tilde{m}^{1/2} \tag{2.2}$$

that is, for x between the turning points $x = \pm 2\tilde{m}^{1/2}$ of the mth-order Hermite function (Olver 1974). On the other hand, the expansions in terms of Airy functions are valid for $\tau > -1$, which includes one turning point, and, by the symmetry of the Hermite functions, the whole range $-\infty < x < \infty$. For simplicity, in spite of this wider range of validity of the Airy expansions, we shall work here with expansions in elementary functions. The expansion of interest here, valid if $m \to \infty$ and $|\tau| < 1$, is (Olver 1959):

$$U(-\tilde{m}, x) \sim b_m (1-\tau^2)^{-1/4} \bigg\{ [1 + \mathrm{O}(\tilde{m}^{-2})] \cos[2\tilde{m}\Delta(\tau) - \pi/4] + \left[\frac{x}{16\tilde{m}^{3/2}} + \mathrm{O}(\tilde{m}^{-5/2}) \right] \sin[2\tilde{m}\Delta(\tau) - \pi/4] \bigg\} \tag{2.3}$$

where

$$b_m \sim \sqrt{2}\, \mathrm{e}^{-\tilde{m}/2} \tilde{m}^{(\tilde{m}-1/2)/2} \left[1 - \frac{1}{48\tilde{m}} + \mathrm{O}(\tilde{m}^{-2}) \right] \tag{2.4}$$

$$\Delta(\tau) = \tfrac{1}{2}[\cos^{-1}\tau - \tau(1-\tau^2)^{1/2}]. \tag{2.5}$$

We have

$$\mathrm{e}^{\mathrm{i}[2\tilde{m}\Delta(\tau) - \pi/4]} = \mathrm{i}^m\, \mathrm{e}^{-\mathrm{i}\rho(\tilde{m}, x)} \tag{2.6}$$

where

$$\rho(\hat{m}, x) = \tilde{m}[\sin^{-1}\tau + \tau(1-\tau^2)^{1/2}] \tag{2.7}$$

and τ is given by (2.2). The limit $m \to \infty$ for a fixed x corresponds to $\tau \to 0$. In this case, we may expand the factor $(1-\tau^2)^{-1/4}$ in (2.3) in powers of τ, and substitute (2.4) and (2.6) into (2.3), to get

$$U(-\tilde{m}, x) \sim 2^{-1/2}\, \mathrm{e}^{-\tilde{m}/2} \tilde{m}^{(\tilde{m}-1/2)/2} \sum_{\mu = \pm 1} \mathrm{i}^{\mu m}\, \mathrm{e}^{-\mathrm{i}\mu\rho(\tilde{m}, x)} \left[1 + \frac{3x^2 - 1}{48\tilde{m}} - \frac{\mathrm{i}\mu x}{16\tilde{m}^{3/2}} + \dots \right]. \tag{2.8}$$

The Stirling approximation for the factorial appearing in (2.1) is (Olver 1959)

$$(m!)^{-1/2} \sim (2\pi)^{-1/4} e^{\tilde{m}/2} \tilde{m}^{-\tilde{m}/2} \left[1 + \frac{1}{48\tilde{m}} + O(\tilde{m}^{-2})\right] \tag{2.9}$$

valid for $m \gg 1$. Using (2.8) and (2.9) in (2.1) we get an asymptotic expansion for the mth-order Hermite function:

$$\psi_m(x/\sqrt{2}) \sim (8\pi^2\tilde{m})^{-1/4} \sum_{\mu=\pm 1} i^{\mu m} e^{-i\mu\rho(\tilde{m},x)} \left[1 + \frac{x^2}{16\tilde{m}} - \frac{i\mu x}{16\tilde{m}^{3/2}} + \dots\right]. \tag{2.10}$$

The above expansion is valid if

$$m \gg 1, x^2/4. \tag{2.11}$$

We note that (2.11) implies that the second and third terms in brackets in (2.10) are respectively much smaller than $\frac{1}{4}$ and $\frac{1}{8}$. Thus, they will always represent small corrections to the leading order behaviour.

Now, to use (2.10) in the derivation of an asymptotic expression for the remainder $R_M(x)$, it is necessary to assume some definite behaviour of the coefficients c_m for large m. In the following we shall suppose that c_m decays algebraically with m as $m \to \infty$,

$$c_m = \tilde{m}^{-\beta} \tag{2.12}$$

for some constant β, and denote $F(x)$ and $R_M(x)$ as $F_\beta(x)$ and $R_M(\beta, x)$, respectively. A slightly more general case, in which c_m has an asymptotic expansion

$$c_m \sim \tilde{m}^{-\beta}(\lambda_0 + \lambda_1\tilde{m}^{-\sigma} + \lambda_2\tilde{m}^{-2\sigma} + \dots) \tag{2.13}$$

with β, λ_j and σ constants, then clearly reduces to the calculation of $R_M(\beta, x)$, $R_M(\beta + \sigma, x)$, etc.

Substituting (2.10) and (2.12) into (1.2), one easily obtains the following asymptotic expansion for the remainder of (1.1):

$$R_M(\beta, x) \sim (2/\pi^2)^{1/4} \operatorname{Re} \sum_{j=0}^{\infty} F_j A_{j+2\beta+1/2} \tag{2.14}$$

where the first F_j are given by

$$F_0 = 1 \qquad F_1 = 0 \qquad F_2 = x^2/16 \qquad F_3 = x/16i \tag{2.15}$$

and A_r denotes the 'auxiliary series'

$$A_r = \sum_{m=M}^{\infty} \tilde{m}^{-r/2} i^m e^{-i\rho(\tilde{m},x)}. \tag{2.16}$$

By (2.11), the expansion (2.14) is valid if

$$M \gg 1, x^2/4. \tag{2.17}$$

From the arguments given following (2.11), we conclude that the series (2.14) may for numerical purposes be truncated at a low value of j. However, depending on the value of β, the series $A_{j+2\beta+1/2}$, whose terms decay in magnitude algebraically with m, may converge very slowly or even diverge. Therefore, to use (2.14) to evaluate the remainder it would be desirable to have a method to compute (2.16) when $M \to \infty$, other than direct term-by-term summation. Such a method is developed in the next section from asymptotic considerations.

3. Auxiliary series

In this section we derive an asymptotic expression for the auxiliary series A_r with the aid of the Poisson summation formula (Henrici 1977). If $f(t)$ is a complex-valued function of the real variable t, then Poisson's formula states that

$$\sum_{m=-\infty}^{\infty} f(m) = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}_n \tag{3.1}$$

where

$$\hat{f}_n = \int_{-\infty}^{\infty} e^{-2\pi i n t} f(t)\,dt \tag{3.2}$$

denotes the Fourier transform of f. Poisson's formula is valid provided f is absolutely integrable, of bounded variation, and for all t_0 satisfies the relation

$$f(t_0) = \tfrac{1}{2} \lim_{t\to t_0+} f(t) + \tfrac{1}{2} \lim_{t\to t_0-} f(t). \tag{3.3}$$

The series A_r is absolutely convergent only for $\mathrm{Re}\, r > 2$. In order to transform A_r by Poisson's formula for an arbitrary value of r, we will interpret the sum of A_r in an Abelian sense (Hardy 1956). Define

$$f(t) = \begin{cases} \tilde{t}^{-r/2} \exp[-\varepsilon\zeta(\tilde{t}) + i\pi t/2 - i\rho(\tilde{t}, x)] & t > M \\ \tfrac{1}{2} i^M \tilde{M}^{-r/2} \exp[-\varepsilon\zeta(\tilde{M}) - i\rho(\tilde{M}, x)] & t = M \\ 0 & t < M \end{cases} \tag{3.4}$$

where $\varepsilon > 0$, $\tilde{t} = t + 1/2$ and the function $\zeta(\tilde{t})$ tends to $+\infty$ as $t \to \infty$. Let us also assume that $\zeta(\tilde{t})$ has a convergent expansion

$$\zeta(\tilde{t}) = \tilde{t}^{1/2}\left(1 + \frac{\zeta_1}{\tilde{t}} + \frac{\zeta_2}{\tilde{t}^2} + \dots\right) \tag{3.5}$$

so that

$$i\rho(\tilde{t}, x) + \varepsilon\zeta(\tilde{t}) = \tilde{t}^{1/2}\left(\gamma + \frac{\gamma_1}{\tilde{t}} + \frac{\gamma_2}{\tilde{t}^2} + \dots\right)$$

$$= \gamma\tilde{t}^{1/2} + i\phi(\varepsilon, \tilde{t}, x) \tag{3.6}$$

where the first coefficients γ_j are

$$\gamma = ix + \varepsilon \qquad \gamma_1 = x^3/24i + \varepsilon\zeta_1 \tag{3.7}$$

and $\phi(\varepsilon, \tilde{t}, x) \to 0$ as $t \to \infty$. The function $f(t)$ satisfies the hypotheses of Poisson's formula, and it follows from (3.1) and (3.4) that

$$A_r = \tfrac{1}{2}\tilde{M}^{-r/2}\mathrm{i}^M \, \mathrm{e}^{-\mathrm{i}\rho(\tilde{M},x)} + \lim_{\varepsilon\to 0^+}\lim_{N\to\infty}\sum_{n=-N}^{N}\hat{f}_n \tag{3.8}$$

where A_r is taken in the sense of the Abelian summability method (A, λ_m), with $\lambda_m = \zeta(\tilde{m})$ (Hardy 1956). In view of (3.5), this method is clearly equivalent to the Abelian method $(A, m^{1/2})$. The summability of A_r by this method can be proved in two steps. First, Hardy (1956 p 141) showed that the series $\Sigma\, m^{-b}\exp[aim^{1/2}]$ (a real and non-zero) is summable by the Cesàro method (C, k) for some k, and hence summable by the Abelian method (A, m); his argument can be readily modified to prove the (A, m) summability of A_r (and also of the series B_r and C_r, to be defined in section 4). Second, a theorem of Cartwright (Hardy 1956 p 381) allows us to deduce $(A, m^{1/2})$ summability from (A, m) summability for A_r, B_r and C_r.

Next, we will derive an asymptotic expansion for $\hat{f}_n$ in inverse powers of $\tilde{M}^{1/2}$, valid in some neighbourhood of $\varepsilon = 0$ and for all integers n. Then, we will obtain from (3.8) an asymptotic expansion for the Abelian sum of A_r, also in inverse powers of $\tilde{M}^{1/2}$.

The Fourier transform of f is given by (note that n, M are integers)

$$\begin{aligned}\hat{f}_n &= \int_M^\infty \tilde{t}^{-r/2}\exp[-2\pi\mathrm{i}\bar{n}t - \varepsilon\zeta(\tilde{t}) - \mathrm{i}\rho(\tilde{t}, x)]\,\mathrm{d}t \\ &= 2(-1)^n\,\mathrm{e}^{-\mathrm{i}\pi/4}\int_{\tilde{M}^{1/2}}^\infty s^{1-r}\exp[-2\pi\mathrm{i}\bar{n}s^2 - \varepsilon\zeta(s^2) - \mathrm{i}\rho(s^2, x)]\,\mathrm{d}s\end{aligned} \tag{3.9}$$

where $\bar{n} = n - 1/4$ and $s = \tilde{t}^{1/2}$. Using (2.17), we may compare the terms in the exponential in (3.9):

$$\begin{aligned}\frac{|\varepsilon\zeta(s^2) + \mathrm{i}\rho(s^2, x)|}{|2\pi\mathrm{i}\bar{n}s^2|} &\leqslant \frac{|x|}{4\pi|\bar{n}|s}\max_{0\leqslant\tau\leqslant 1}\left(\frac{\sin^{-1}\tau}{\tau} + (1-\tau^2)^{1/2}\right) + \mathrm{O}(\varepsilon) \\ &\leqslant \frac{2|x|}{\pi\tilde{M}^{1/2}} + \mathrm{O}(\varepsilon) \ll \frac{4}{\pi} + \mathrm{O}(\varepsilon).\end{aligned} \tag{3.10}$$

Thus, for small ε, $-2\pi\mathrm{i}\bar{n}s^2$ is the dominant term in the exponential in (3.9), for all $s \geqslant \tilde{M}^{1/2}$ and for every integer n. This indicates that an asymptotic expansion for $\hat{f}_n$ may be obtained by integration by parts (Olver 1974). To this end, let us define the variable

$$z = z(s) = 2\bar{n}^{1/2}s + \gamma/2\pi\mathrm{i}\bar{n}^{1/2}. \tag{3.11}$$

The integral (3.9) then becomes

$$\hat{f}_n = 2(-1)^n\exp[-\mathrm{i}\pi/4 - \mathrm{i}\gamma^2/8\pi\bar{n}]\int_{\tilde{M}^{1/2}}^\infty s^{1-r}\exp[-\mathrm{i}\pi z(s)^2/2 - \mathrm{i}\phi(\varepsilon, s^2, x)]\,\mathrm{d}s. \tag{3.12}$$

To integrate (3.12) by parts, we will make use of the known Fresnel integral (Abramowitz and Stegun 1965)

$$\int \mathrm{e}^{-\mathrm{i}\pi z^2/2}\,\mathrm{d}z = \mathrm{i}Q(z)\,\mathrm{e}^{-\mathrm{i}\pi z^2/2} \tag{3.13}$$

where $Q(z)$ is a function with the following asymptotic behaviour as $z\to\infty$ with $|\arg z| \leqslant \pi/2$:

$$Q(z) \sim (\pi z)^{-1}[1 + \mathrm{i}/\pi z^2 - 3/(\pi z^2)^2 + \dots]. \tag{3.14}$$

Integrating by parts repeatedly, with the aid of (3.13), one obtains the formal expansion

$$\int_{\tilde{M}^{1/2}}^{\infty} s^{1-r} \exp[-\mathrm{i}\pi z(s)^2/2 - \mathrm{i}\phi(\varepsilon, s^2, x)]\,\mathrm{d}s$$

$$\sim \frac{\mathrm{i}Q(z(s)) \exp[-\mathrm{i}\pi z(s)^2/2 - \mathrm{i}\phi(\varepsilon, s^2, x)]}{2\bar{n}^{1/2} s^{r-1}}$$

$$\times\left\{1 + \frac{\mathrm{i}Q(z(s))}{2\bar{n}^{1/2}}\left(\mathrm{i}\phi'(\varepsilon, s^2, x) + \frac{r-1}{s}\right)\right.$$

$$-\frac{Q(z(s))}{4\bar{n}}\left[Q(z(s))\left(\left(\mathrm{i}\phi'(\varepsilon, s^2, x) + \frac{r-1}{s}\right)^2\right.\right.$$

$$\left.-\left(\mathrm{i}\phi''(\varepsilon, s^2, x) - \frac{r-1}{s^2}\right)\right)$$

$$\left.\left.-2\bar{n}^{1/2}\frac{\mathrm{d}Q}{\mathrm{d}z}(z(s))\left(\mathrm{i}\phi'(\varepsilon, s^2, x) + \frac{r-1}{s}\right)\right] + \ldots\right\}\Bigg|_{s=\tilde{M}^{1/2}}^{s=+\infty} \qquad (3.15)$$

where $\phi'(\varepsilon, s^2, x) = (\partial/\partial s)\phi(\varepsilon, s^2, x)$ and $\phi''(\varepsilon, s^2, x) = (\partial^2/\partial s^2)\phi(\varepsilon, s^2, x)$. Since $\mathrm{Re}(-\mathrm{i}\pi z(s)^2/2) \sim -\mathrm{Re}(\gamma s) = -\varepsilon s \to -\infty$ as $s \to \infty$, the right-hand side of (3.15) vanishes at the upper limit of integration; at $s = \tilde{M}^{1/2}$, the expansion can be simplified by using (3.6), (3.11) and (3.14), and expanding the inverse powers of $z(\tilde{M}^{1/2})$ in inverse powers of $\tilde{M}^{1/2}$. The expansion (3.14) can be used with $z = z(\tilde{M}^{1/2})$ because

$$|z(\tilde{M}^{1/2})| \geqslant 2|\bar{n}|^{1/2}\tilde{M}^{1/2}\left[1 - \frac{|\gamma/2\pi\mathrm{i}\bar{n}^{1/2}|}{2|\bar{n}|^{1/2}\tilde{M}^{1/2}}\right]$$

$$\geqslant 2|\bar{n}|^{1/2}\tilde{M}^{1/2}\left[1 - \frac{|x|}{4\pi|\bar{n}|\tilde{M}^{1/2}} + \mathrm{O}(\varepsilon)\right]$$

$$\gg 1 - \frac{2}{\pi} + \mathrm{O}(\varepsilon) \qquad (3.16)$$

and we can ensure that $|\arg z(\tilde{M}^{1/2})| \leqslant \pi/2$ by choosing $\bar{n}^{1/2} = -\mathrm{i}|\bar{n}|^{1/2}$ for $\bar{n} < 0$. With these transformations, (3.15) and (3.12) yield the following expansion for $\hat{f}_n$:

$$\hat{f}_n \sim \frac{\mathrm{i}^{M-1}\exp[-\varepsilon\zeta(\tilde{M}) - \mathrm{i}\rho(\tilde{M}, x)]}{2\pi\bar{n}\tilde{M}^{r/2}}\left\{1 + \frac{\mathrm{i}\gamma}{4\pi\bar{n}\tilde{M}^{1/2}} + \frac{1}{4\pi\bar{n}}\left(\mathrm{i}r - \frac{\gamma^2}{4\pi\bar{n}}\right)\frac{1}{\tilde{M}}\right.$$

$$-\frac{1}{4\pi\bar{n}}\left[\frac{\gamma}{4\pi\bar{n}}\left(\frac{\mathrm{i}\gamma^2}{4\pi\bar{n}} + 2r + 1\right) + \mathrm{i}\gamma_1\right]\frac{1}{\tilde{M}^{3/2}} + \frac{1}{(4\pi\bar{n})^2}$$

$$\left.\times\left[\frac{\gamma^4}{(4\pi\bar{n})^2} - \frac{3(r+1)\mathrm{i}\gamma^2}{4\pi\bar{n}} - (r^2+2) + 2\gamma_1\gamma\right]\frac{1}{\tilde{M}^2} + \ldots\right\}. \qquad (3.17)$$

The right-hand side of (3.17) can be summed over n by using the identity (Henrici 1977)

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{(n-w)^{k+1}} = -\pi\frac{\mathrm{d}^k}{\mathrm{d}w^k}\cot\pi w \qquad k = 0, 1, 2, \ldots \qquad (3.18)$$

with $w=1/4$. Poisson's formula then yields

$$\sum_{m=M}^{\infty} \frac{\mathrm{i}^m \exp[-\varepsilon\zeta(\tilde{m})-\mathrm{i}\rho(\tilde{m},x)]}{\tilde{m}^{r/2}}$$

$$\sim \frac{\mathrm{i}^M \exp[-\varepsilon\zeta(\tilde{M})-\mathrm{i}\rho(\tilde{M},x)]}{2\tilde{M}^{r/2}}\left\{(1+\mathrm{i})+\frac{\gamma}{2\tilde{M}^{1/2}}+\frac{1}{2}\left(r-\frac{1}{4}\mathrm{i}\gamma^2\right)\frac{1}{\tilde{M}}\right.$$

$$-\frac{1}{2}\left[\frac{1}{4}\gamma\left(\frac{1}{3}\gamma^2+(2r+1)\mathrm{i}\right)+\gamma_1\right]\frac{1}{\tilde{M}^{3/2}}$$

$$\left.+\frac{1}{8}\left[\frac{5\mathrm{i}}{48}\gamma^4-(r+1)\gamma^2-\mathrm{i}[(r^2+2)-2\gamma_1\gamma]\right]\frac{1}{\tilde{M}^2}+\ldots\right\}. \tag{3.19}$$

Setting $\varepsilon=0$ in this expression, we obtain with the aid of (3.7) the asymptotic expansion for the Abelian sum of A_r that we are seeking:

$$A_r \sim \frac{1}{2}\tilde{M}^{-r/2}\mathrm{i}^M \mathrm{e}^{-\mathrm{i}\rho(\tilde{M},x)}\left\{(1+\mathrm{i})+\tfrac{1}{2}\mathrm{i}x\tilde{M}^{-1/2}+\tfrac{1}{2}(r+\tfrac{1}{4}\mathrm{i}x^2)\tilde{M}^{-1}\right.$$

$$+\frac{x}{8}\left[(2r+1)+\frac{1}{2}\mathrm{i}x^2\right]\tilde{M}^{-3/2}$$

$$\left.+\frac{1}{8}\left[\frac{9\mathrm{i}}{48}x^4+(r+1)x^2-\mathrm{i}(r^2+2)\right]\tilde{M}^{-2}+\ldots\right\}. \tag{3.20}$$

This result can also be obtained following Wimp's method by substituting

$$\tilde{a}_M=\tilde{M}^{-r/2}\mathrm{i}^M \mathrm{e}^{-\mathrm{i}\rho(\tilde{M},x)} \qquad \tilde{R}_M=\tilde{M}^{-\theta}\mathrm{i}^M \mathrm{e}^{-\mathrm{i}\rho(\tilde{M},x)}(K_0+K_1\tilde{M}^{-1/2}+\ldots) \tag{3.21}$$

Table 1. Computation of A_3 for $x=8$ and $M=48$, using (3.19) with $\varepsilon=0$. Successive rows in the table give the value of the expansion (3.19) truncated at order $\tilde{M}^{-k/2}$, $k=0,1,2,3,4$.

k	Value of (3.19)
0	0.000 301 − 0.002 072i
1	0.000 983 − 0.002 580i
2	0.001 151 − 0.002 763i
3	0.001 245 − 0.002 871i
4	0.001 281 − 0.002 923i

Table 2. Asymptotic evaluation of $F_\beta(x)$, for (a) $\beta=3/4$, $x=10$ and (b) $\beta=0$, $x=4$. See text for the meaning of each row.

	(a)		(b)	
M	75	85	12	16
Partial sum	0.2294	0.2466	1.262	1.824
$j=0,1$	0.2370	0.2389	1.629	1.693
$j=2$	0.2376	0.2383	1.660	1.684
$j=3$	0.2376	0.2383	1.660	1.682

in (1.6) and finding the constants θ and K_n. Wimp's original formulation prescribed the use of the first-order term $\tilde{M}^{1/2}x$ instead of $\rho(\tilde{M}, x)$ in (3.21). However, it can be seen from (3.6) that this would introduce terms $O(x^3/\tilde{M})$ in the expansion for A_r: such terms may become large if we assume only condition (2.17). Hence the resulting expansion would converge more slowly than (3.20).

A numerical example of the use of (3.19), with $r=3$, is given in table 1. In this example, note that the chosen truncation ($M=48$ for $x=8$) is consistent with (2.17). For comparison, the value obtained by summing (2.16) directly up to $m=1233$ (such a high value of m is necessary, due to the slow convergence of the series) is $A_3 \approx 0.001\,294-0.002\,988\mathrm{i}$. In table 2, (3.19) is used in (2.14) to compute some values of $F_\beta(x)$. The first row of the table gives the value of (1.1) truncated at $m=M-1$, and the remaining rows give the effect of successively adding the terms $j=0, 1, 2, 3$ of (2.14). To check the correctness of the computation, we compute $F_\beta(x)$ for two different values of M which are consistent with (2.17). In table 2(*a*) $\beta=3/4$, and the leading order term in (2.14) will involve the series A_2, whose general term is $O(m^{-1}) = O\{(\lambda_m-\lambda_{m-1})/\lambda_m\}$ (recall that $\lambda_m=\zeta(\tilde{m})=O(m^{1/2})$). Since A_2 is summable (A, λ_m), this implies that A_2 (and hence $F_{3/4}(x)$) is conditionally convergent (Hardy 1956 p 161). An application to a divergent Hermite series, $F_0(x)$, is shown in table 2(*b*). Numerical experiments with (2.14) and (3.19) indicate that $F_\beta(x)$ can be computed with an error of a few per cent by taking $M \geqslant 3\max\{1, x^2/4\}$.

4. Asymptotic expansion for the remainder of $G(x, y)$

Series of the form (1.4) may be treated by asymptotic techniques similar to those developed in the two previous sections. For definiteness, let us consider again the case where c_m is given by (2.12), and let us write $G_\beta(x, y)$ for $G(x, y)$ and $Q_M(\beta, x, y)$ for its Mth remainder. Using (2.10) in (1.4), one finds an asymptotic expansion for $Q_M(\beta, x, y)$:

$$Q_M(\beta, x, y) \sim (2\pi^2)^{-1/2}\,\mathrm{Re}\sum_{j=0}^{\infty}(G_j B_{j+2\beta+1} + H_j C_{j+2\beta+1}). \tag{4.1}$$

Here the first G_j and H_j are

$$\begin{aligned} &G_0 = H_0 = 1 \qquad G_1 = H_1 = 0 \qquad G_2 = H_2 = (x^2+y^2)/16 \\ &G_3 = (x+y)/16\mathrm{i} \qquad H_3 = (x-y)/16\mathrm{i} \end{aligned} \tag{4.2}$$

and B_r and C_r denote the 'auxiliary series'

$$B_r = \sum_{m=M}^{\infty} \tilde{m}^{-r/2}(-1)^m \exp\{-\mathrm{i}[\rho(\tilde{m}, x) + \rho(\tilde{m}, y)]\} \tag{4.3}$$

$$C_r = \sum_{m=M}^{\infty} \tilde{m}^{-r/2} \exp\{-\mathrm{i}[\rho(\tilde{m}, x) - \rho(\tilde{m}, y)]\}. \tag{4.4}$$

By (2.11), the expansion (4.1) is valid for

$$M \gg 1, x^2/4, y^2/4. \tag{4.5}$$

To derive an asymptotic expansion for B_r, analogous to (3.19), one can proceed in complete analogy with the treatment given for the series A_r. Defining a function

$g(t)$ by

$$g(t)=\begin{cases}\tilde{t}^{-r/2}\exp\{-\varepsilon\zeta(\tilde{t})+\mathrm{i}\pi t-\mathrm{i}[\rho(\tilde{t},x)+\rho(\tilde{t},y)]\} & t>M\\ \frac{1}{2}(-1)^M\tilde{M}^{-r/2}\exp\{-\varepsilon\zeta(\tilde{M})-\mathrm{i}[\rho(\tilde{M},x)+\rho(\tilde{M},y)]\} & t=M\\ 0 & t<M\end{cases} \tag{4.6}$$

with $\varepsilon>0$ and $\zeta(\tilde{t})$ given by (3.5), we will have

$$\begin{aligned}\mathrm{i}[\rho(\tilde{t},x)+\rho(\tilde{t},y)]+\varepsilon\zeta(\tilde{t}) &= \tilde{t}^{1/2}\left(\eta+\frac{\eta_1}{\tilde{t}}+\frac{\eta_2}{\tilde{t}^2}+\ldots\right)\\ &= \eta\tilde{t}^{1/2}+\mathrm{i}\theta(\varepsilon,\tilde{t},x,y)\end{aligned} \tag{4.7}$$

where the first coefficients η_j are

$$\eta=\mathrm{i}(x+y)+\varepsilon \qquad \eta_1=(x^3+y^3)/24\mathrm{i}+\varepsilon\zeta_1 \tag{4.8}$$

and $\theta(\varepsilon,\tilde{t},x,y)\to 0$ as $t\to\infty$. Applying Poisson's formula to the function g, we can modify slightly the asymptotic arguments of section 3 to get the result

$$\begin{aligned}\sum_{m=M}^{\infty}&\frac{(-1)^m\exp\{-\varepsilon\zeta(\tilde{m})-\mathrm{i}[\rho(\tilde{m},x)+\rho(\tilde{m},y)]\}}{\tilde{m}^{r/2}}\\ &\sim\frac{(-1)^M\exp\{-\varepsilon\zeta(\tilde{M})-\mathrm{i}[\rho(\tilde{M},x)+\rho(\tilde{M},y)]\}}{2\tilde{M}^{r/2}}\\ &\quad\times\left\{1+\frac{\eta}{4\tilde{M}^{1/2}}+\frac{r}{4\tilde{M}}-\frac{1}{4}\left(\frac{1}{48}\eta^3+\eta_1\right)\frac{1}{\tilde{M}^{3/2}}-\frac{(r+1)\eta^2}{64\tilde{M}^2}+\ldots\right\}.\end{aligned} \tag{4.9}$$

When $\varepsilon=0$, the above expansion is asymptotic to the (A,λ_m) sum of B_r. In table 3, this expansion is employed to evaluate B_3 for $x=6$, $y=4$, $M=27$. The direct summation of (4.3) up to $m=1098$ gives the approximate value $B_3\approx -0.003\,265-0.002\,419\mathrm{i}$.

Table 3. Computation of B_3 for $x=6$, $y=4$, $M=27$, using (4.9) with $\varepsilon=0$. The meaning of each row is the same as in table 1.

k	Value of (4.9)
0	−0.003 424 − 0.000 544i
1	−0.003 165 − 0.002 176i
2	−0.003 258 − 0.002 191i
3	−0.003 228 − 0.002 384i
4	−0.003 256 − 0.002 389i

The auxiliary series C_r can also be evaluated with the Poisson summation formula, but the argument here is more subtle, as we shall see shortly. Defining the function $h(t)$ by

$$h(t)=\begin{cases}\tilde{t}^{-r/2}\exp\{-\varepsilon\zeta(\tilde{t})-\mathrm{i}[\rho(\tilde{t},x)-\rho(\tilde{t},y)]\} & t>M\\ \frac{1}{2}\tilde{M}^{-r/2}\exp\{-\varepsilon\zeta(\tilde{M})-\mathrm{i}[\rho(\tilde{M},x)-\rho(\tilde{M},y)]\} & t=M\\ 0 & t<M\end{cases} \tag{4.10}$$

with $\varepsilon > 0$ and $\zeta(\tilde{t})$ again given by (3.5), we can write

$$\mathrm{i}[\rho(\tilde{t}, x) - \rho(\tilde{t}, y)] + \varepsilon\zeta(\tilde{t}) = \tilde{t}^{1/2}\left(\xi + \frac{\xi_1}{\tilde{t}} + \frac{\xi_2}{\tilde{t}^2} + \ldots\right)$$
$$= \xi\tilde{t}^{1/2} + \mathrm{i}\omega(\varepsilon, \tilde{t}, x, y) \tag{4.11}$$

where the first coefficients ξ_j are

$$\xi = \mathrm{i}(x - y) + \varepsilon \qquad \xi_1 = (x^3 - y^3)/24\mathrm{i} + \varepsilon\zeta_1 \tag{4.12}$$

and $\omega(\varepsilon, \tilde{t}, x, y) \to 0$ as $t \to \infty$.

The series

$$\sum_{m=M}^{\infty} \tilde{m}^{-r/2} \exp\{-\varepsilon\zeta(\tilde{m}) - \mathrm{i}[\rho(\tilde{m}, x) - \rho(\tilde{m}, y)]\}$$
$$= \sum_{m=M}^{\infty} \tilde{m}^{-r/2} \exp[-\xi\tilde{m}^{1/2} - \mathrm{i}\omega(\varepsilon, \tilde{m}, x, y)]$$
$$= \sum_{m=M}^{\infty} d_m \exp(-\xi\tilde{m}^{1/2}) = D(\xi) \tag{4.13}$$

can be regarded as a Dirichlet series, whose abscissa of convergence δ is obviously zero. Since $\arg d_m \to 0$ as $m \to \infty$, it follows that $\xi = \delta = 0$ is a point of non-continuability of the Taylor expansion of $D(\xi)$ with centre at any point in the half-plane $\mathrm{Re}\,\xi > 0$ (Saks and Zygmund 1971). This implies that the Abelian sum of C_r (and therefore that of $G_\beta(x, y)$) may be non-analytic at $x = y$. This is in contrast to the series B_r, which by (4.9) is an analytic function of (x, y) for all r. In fact, we shall see below that, for all r, C_r is analytic everywhere except for $x = y$.

Proceeding with the asymptotic calculation of C_r, the Fourier transform of $h(t)$ can be written as

$$\hat{h}_n = 2(-1)^n \int_{\tilde{M}^{1/2}}^{\infty} s^{1-r} \exp\{-2\pi\mathrm{i}ns^2 - \varepsilon\zeta(s^2) - \mathrm{i}[\rho(s^2, x) - \rho(s^2, y)]\}\,\mathrm{d}s \tag{4.14}$$

where $s = \tilde{t}^{1/2}$. Now, it can be verified using (3.10) that for small ε the term $-2\pi\mathrm{i}ns^2$ will be dominant in the exponential in (4.14), but only for $n \neq 0$. In this case, one can use integration by parts to determine the asymptotic behaviour of $\hat{h}_n$, exactly as in section 3. The result is an expansion identical to (3.17), wherein the leading factor i^M is dropped, and the replacements

$$[\rho(\tilde{M}, x), \bar{n}, \gamma, \gamma_1, \ldots] \to [\rho(\tilde{M}, x) - \rho(\tilde{M}, y), n, \xi, \xi_1, \ldots]$$

are made. Using the well known sums

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} n^{-4} = \frac{\pi^4}{90} \tag{4.15}$$

one easily verifies that

$$\lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \hat{h}_n \sim \frac{\exp\{-\varepsilon\zeta(\tilde{M}) - \mathrm{i}[\rho(\tilde{M}, x) - \rho(\tilde{M}, y)]\}}{24\tilde{M}^{r/2}}$$
$$\times \left\{\frac{\xi}{\tilde{M}^{1/2}} + \frac{r}{\tilde{M}} - \left(\frac{1}{240}\xi^3 + \xi_1\right)\frac{1}{\tilde{M}^{3/2}} - \frac{(r+1)\xi^2}{80\tilde{M}^2} + \ldots\right\}. \tag{4.16}$$

The term

$$\hat{h}_0 = 2\int_{\tilde{M}^{1/2}}^{\infty} s^{1-r}\exp[-\xi s - \mathrm{i}\omega(\varepsilon, s^2, x, y)]\,\mathrm{d}s \tag{4.17}$$

will give rise to the non-analytic behaviour of C_r at $x = y$, and must be treated separately. If we take (4.5) to mean

$$M \geq K\max\{1, x^2/4, y^2/4\} \tag{4.18}$$

with $K > 1$ a constant, then for $t \geq M$ we have the estimate

$$\begin{aligned}|\omega(0, \tilde{t}, x, y)| &\leq |x-y| \max_{x\leq w\leq y}\left|\frac{\partial\phi}{\partial w}(0, \tilde{t}, w)\right| \\ &\leq |x-y|\frac{\max\{x^2, y^2\}}{4\tilde{t}^{1/2}} \max_{0<\tau<K^{-1/2}} \tau^{-2}[1-(1-\tau^2)^{1/2}] \\ &\leq [1-(1-K^{-1})^{1/2}]|X|\end{aligned} \tag{4.19}$$

where

$$X = \tilde{M}^{1/2}\xi. \tag{4.20}$$

Thus, for $|X|$ bounded we can evaluate $\hat{h}_0$ numerically from the convergent expansion

$$\hat{h}_0 = 2\int_{\tilde{M}^{1/2}}^{\infty} \mathrm{e}^{-\xi s}\sum_{n=0}^{\infty} q_n s^{-n-r+1}\,\mathrm{d}s = 2\sum_{n=0}^{\infty} q_n \xi^{n+r-2}\Gamma(2-n-r, X) \tag{4.21}$$

where $\Gamma(a, X)$ denotes the incomplete gamma function (Abramowitz and Stegun 1965) and the q_n are the coefficients in the expansion of $\mathrm{e}^{-\mathrm{i}\omega}$ in powers of s^{-1} (see (4.11)), listed in the appendix. From the properties of $\Gamma(a, X)$, it can be shown that each term of (4.21) is non-analytic at $\xi = 0$, in accordance with a previous remark in this section. For simplicity, we will discuss only the case where 2β is an integer, so that by (4.1) we need only consider the C_r for integer r. To evaluate (4.21), we may then note that $\Gamma(a, X)$ is given in terms of elementary functions for $a = 1, 2, \ldots$; that $\Gamma(0, X) = E_1(X)$, the exponential integral; and that the $\Gamma(a, X)$, $a = -1, -2, \ldots$, may be obtained from $\Gamma(0, X)$ by the recurrence relation

$$\Gamma(a-1, X) = (a-1)^{-1}[\Gamma(a, X) - X^{a-1}\,\mathrm{e}^{-X}]. \tag{4.22}$$

The exponential integral is easily evaluated with a computer (see the appendix). Based on numerical experiments with (4.21), we have verified that we can compute $\hat{h}_0$ accurate to a few per cent for $|X| \leq 25$ when the expansion is truncated at $n = 10$.

To compute $\hat{h}_0$ when X is large, we may use the following asymptotic expansion, which follows from (4.17) by iterated integration by parts (Olver 1974):

$$\hat{h}_0 \sim 2\,\mathrm{e}^{-X}\sum_{n=0}^{\infty}\xi^{-n-1}\frac{\partial^n}{\partial s^n}\frac{\exp[-\mathrm{i}\omega(\varepsilon, s^2, x, y)]}{s^{r-1}}\bigg|_{s=\tilde{M}^{1/2}}. \tag{4.23}$$

When $|X| = 25$, we have verified the numerical agreement between this expansion truncated at $n = 3$ and (4.21) truncated at $n = 10$. For $|X| > 25$, $\hat{h}_0$ can be computed with an accuracy of a few per cent by using (4.23) with the indicated truncation.

Substituting (4.10) and (4.16) in Poisson's formula, we get finally

$$\sum_{m=M}^{\infty} \frac{\exp\{-\varepsilon\zeta(\tilde{m}) - \mathrm{i}[\rho(\tilde{m}, x) - \rho(\tilde{m}, y)]\}}{\tilde{m}^{r/2}}$$
$$\sim \hat{h}_0 + \frac{\exp\{-\varepsilon\zeta(\tilde{M}) - \mathrm{i}[\rho(\tilde{M}, x) - \rho(\tilde{M}, y)]\}}{2\tilde{M}^{r/2}}$$
$$\times \left\{1 + \frac{\xi}{12\tilde{M}^{1/2}} + \frac{r}{12\tilde{M}} - \frac{1}{12}\left(\frac{1}{240}\xi^3 + \xi_1\right)\frac{1}{\tilde{M}^{3/2}} - \frac{(r+1)\xi^2}{960\tilde{M}^2} + \ldots\right\} \tag{4.24}$$

where $\hat{h}_0$ may be computed from (4.21) or (4.23), depending on the value of $|X|$. The Abelian sum C_r is then given asymptotically by (4.24) with $\varepsilon = 0$.

We can use (4.21) and (4.24) to find the behaviour of $G_\beta(x, y)$ as $x \to y$. For instance, when $\beta = 1/2$, we have

$$Q_M(1/2, x, y) \sim \frac{1}{\pi\sqrt{2}} \operatorname{Re} C_2 + \mathrm{O}(1) \sim \frac{\sqrt{2}}{\pi} \operatorname{Re} \Gamma(0, X) + \mathrm{O}(1)$$
$$\sim -\frac{\sqrt{2}}{\pi} \operatorname{Re} \ln X + \mathrm{O}(1) \sim -\frac{\sqrt{2}}{\pi} \ln|x - y| + \mathrm{O}(1) \tag{4.25}$$

because $\Gamma(0, X) = E_1(X)$ behaves as $-\ln X$ as $X \to 0$. Thus, we conclude that $G_{1/2}(x, y)$ has a logarithmic singularity at $x = y$.

We may compare (4.21), (4.23) and (4.24) with the expansion that would result from the application of Wimp's method to the series C_r. Setting

$$\begin{aligned} \tilde{a}_M &= \tilde{M}^{-r/2} \exp\{-\mathrm{i}[\rho(\tilde{M}, x) - \rho(\tilde{M}, y)]\} \\ \tilde{R}_M &= \tilde{M}^{-\theta'} \exp\{-\mathrm{i}[\rho(\tilde{M}, x) - \rho(\tilde{M}, y)]\}(K'_0 + K'_1\tilde{M}^{-1/2} + \ldots) \end{aligned} \tag{4.26}$$

in (1.6) and determining θ' and K'_n, it is easy to see from (4.11) that $\theta' = (r-1)/2$ and that K'_n will involve terms proportional to $(x-y)^{-n-1}$. In fact, Wimp's method yields (4.24) with $\hat{h}_0$ given by (4.23), and thus it fails to describe the behaviour of C_r (and hence of $G_\beta(x, y)$) as $x \to y$, which is given by (4.21) and (4.24).

Table 4 illustrates the numerical use of (4.24) with $r = 3$, $x = 3$, $y = -2$ and $M = 7$. For comparison, the direct summation of (4.4) up to $m = 6201$ gives $C_3 \approx 0.001\,81 - 0.055\,84\mathrm{i}$. In table 5 we use (4.1), (4.9) and (4.24) to compute $G_{1/2}(x, y)$ for $x = 1$ and $y = -1/2$. The first row of the table gives the Mth partial sum of (1.4), and the other rows give the effect of adding the terms $j = 0, 1, 2, 3$ of (4.1). Just as was done for the example of table 2(*a*), we can argue that $G_{1/2}(x, y)$ is conditionally convergent. Numerical tests of (4.24) indicate that the constant K in (4.18) should be at least 3 in order to produce results with a relative error of a few per cent.

Table 4. Computation of C_3 for $x = 3$, $y = -2$, $M = 7$, using (4.24) with $\varepsilon = 0$. The first row gives the value of $\hat{h}_0$, and the meaning of the other rows is as in table 1.

k	Value of (4.24)
	$\hat{h}_0 = -0.021\,46 - 0.045\,48\mathrm{i}$
0	$-0.000\,97 - 0.058\,63\mathrm{i}$
1	$0.001\,03 - 0.055\,52\mathrm{i}$
2	$0.001\,71 - 0.055\,96\mathrm{i}$
3	$0.001\,82 - 0.055\,79\mathrm{i}$
4	$0.001\,86 - 0.055\,81\mathrm{i}$

Table 5. Asymptotic evaluation of $G_{1/2}(1, -1/2)$, using two different truncations. See text for the meaning of each row.

M	3	10
Partial sum	0.4152	0.1835
$j=0,1$	0.2752	0.2726
$j=2$	0.2718	0.2731
$j=3$	0.2722	0.2732

The main restriction on the use of (4.1) to the evaluation of $G_\beta(x, y)$ occurs when $|x-y|\to\infty$. In this case, $G_\beta(x, y)$ tends to zero very rapidly, while the series (1.4) may diverge or converge slowly. Therefore, the Mth partial sum of (1.4) (and also the remainder $Q_M(\beta, x, y)$) will be much larger than the sum itself, resulting in a large relative error in the computed $G_\beta(x, y)$ due to cancellation. Of course, this effect is inherent in any method of summation which evaluates $G_\beta(x, y)$ as a partial sum plus an approximate remainder. Thus, a computed $G_\beta(x, y)$ which is less than a few per cent of the Mth partial sum of (1.4) is likely to be affected by cancellation, since our asymptotic expansions usually give B_r and C_r accurate to a few per cent. This problem does not occur with the series (2.14), since $F_\beta(y)$ decays or grows more slowly than $G_\beta(x, y)$ tends to zero.

The cancellation effect is more severe for the series $G_\beta(x, y)$ with $\beta=0, -1, -2, \ldots$, because the completeness of the Hermite functions implies that $G_0(x, y)=\sqrt{2}\delta(x-y)$, and $G_\beta(x, y)$ satisfies the inhomogeneous parabolic cylinder equation (Olver 1974)

$$\frac{\partial^2 G_\beta}{\partial x^2}-\frac{1}{4}x^2 G_\beta=-G_{\beta-1}. \tag{4.27}$$

Thus, $G_\beta(x, y)=0$ for $x\neq y$ and $\beta=0, -1, -2, \ldots$, so that asymptotic summation is not needed in these cases. When $\beta=n=1, 2, \ldots$ we can give a closed-form expression for $G_n(x, y)$, which turns out to be too cumbersome for numerical use when $n>1$. Consider the meromorphic function

$$p_n(w)=(-w)^{-n}\Gamma(w+1/2)\,U(w, x)\,U(w, -y) \tag{4.28}$$

which has poles at $w=0$ and $w=-(m+1/2)$, $m=0, 1, 2, \ldots$. The asymptotic behaviour of $p_n(w)$ for large w can be shown to be (Olver 1959)

$$\begin{aligned} p_n(w) &\sim (-1)^n(\pi/2)^{1/2}w^{-n-1/2}\exp[-(x-y)w^{1/2}] \qquad |\arg w|<\pi/2 \\ p_n(w) &\sim (\pi/8)^{1/2}(-w)^{-n-1/2} \\ &\quad\times\frac{\cos[\pi(w+1/2)/2+(-w)^{1/2}x]\cos[\pi(w+1/2)/2-(-w)^{1/2}y]}{\sin[\pi(w+1/2)]} \\ &\quad |\arg(-w)|<\pi/2. \end{aligned} \tag{4.29}$$

If $x>y$, we can then assert that the sum of the residues of $p_n(w)$ vanishes, which yields immediately an expression for $G_n(x, y)$:

$$G_n(x, y)=\frac{(-1)^{n-1}}{\sqrt{\pi}\Gamma(n)}\frac{\mathrm{d}^{n-1}}{\mathrm{d}w^{n-1}}[\Gamma(w+1/2)\,U(w, x)\,U(w, -y)]\bigg|_{w=0}. \tag{4.30}$$

When $x \to \infty$ and y is held fixed, we have $U(w, x) \sim x^{-w-1/2} e^{-x^2/4}[1 + O(x^{-2})]$ (Abramowitz and Stegun 1965), which implies that

$$G_n(x, y) \sim \frac{(-1)^{n-1}}{\sqrt{\pi}\Gamma(n)} \Gamma(1/2) U(0, -y) \frac{d^{n-1}}{dw^{n-1}} U(w, x)\bigg|_{w=0}$$

$$\sim \frac{U(0, -y)}{\Gamma(n)} (\ln x)^{n-1} x^{-1/2} e^{-x^2/4}. \tag{4.31}$$

This expression, though not accurate enough for numerical use, indicates that $G_n(x, y)$ tends to zero faster than any Hermite function $\psi_m(x/\sqrt{2})$ as $x \to \infty$, so that it is not a surprise that cancellation should take place in this limit.

Finally, we wish to comment that the methods of this paper may be used to evaluate any derivative of the series $F(x)$ and $G(x, y)$. In fact, the derivative of the Hermite function of order m has the asymptotic expansion

$$\psi'_m(x/\sqrt{2}) \sim \frac{1}{i}\left(\frac{\tilde{m}}{2\pi^2}\right)^{1/4} \sum_{\mu=\pm 1} \mu i^{\mu m} e^{-i\mu\rho(\tilde{m}, x)} \left[1 - \frac{x^2}{16\tilde{m}} + \frac{i\mu x}{16\tilde{m}^{3/2}} + \ldots\right] \tag{4.32}$$

which follows by differentiation of (2.10). From this expansion and (2.10) one can derive, as in section 2 and in this section, expansions for the remainders of dF_β/dx, $\partial G_\beta/\partial x$, etc. in terms of the auxiliary series A_r, B_r and C_r.

Appendix

Here we give expressions for the first coefficients q_n which appear in (4.21), and discuss a simple numerical procedure for the computation of the exponential integral $E_1(X)$.

The coefficients q_n are defined by the relation

$$\exp[-i\omega(\varepsilon, s^2, x, y)] = \sum_{n=0}^{\infty} q_n s^{-n} \tag{A.1}$$

where $s = \tilde{t}^{1/2}$. In view of (4.11), we may express $q_1, q_2, \ldots$ in terms of $\xi_1, \xi_2, \ldots$:

$$q_0 = 1 \qquad q_1 = -\xi_1 \qquad q_2 = \tfrac{1}{2}\xi_1^2 \qquad q_3 = -(\xi_2 + \tfrac{1}{6}\xi_1^3)$$

$$q_4 = \xi_1(\xi_2 + \tfrac{1}{24}\xi_1^3) \qquad q_5 = -(\xi_3 + \tfrac{1}{2}\xi_1^2\xi_2 + \tfrac{1}{120}\xi_1^5)$$

$$q_6 = \xi_1\xi_3 + \tfrac{1}{2}\xi_2^2 + \tfrac{1}{6}\xi_1^3\xi_2 + \tfrac{1}{720}\xi_1^6$$

$$q_7 = -(\xi_4 + \tfrac{1}{2}\xi_1\xi_2^2 + \tfrac{1}{2}\xi_1^2\xi_3 + \tfrac{1}{24}\xi_1^4\xi_2 + \tfrac{1}{5040}\xi_1^7)$$

$$q_8 = \xi_1\xi_4 + \xi_2\xi_3 + \tfrac{1}{4}\xi_1^2\xi_2^2 + \tfrac{1}{6}\xi_1^3\xi_3 + \tfrac{1}{120}\xi_1^5\xi_2 + \tfrac{1}{40\,320}\xi_1^8$$

$$q_9 = -(\xi_5 + \xi_1\xi_2\xi_3 + \tfrac{1}{2}\xi_1^2\xi_4 + \tfrac{1}{6}\xi_2^3 + \tfrac{1}{12}\xi_1^3\xi_2^2 + \tfrac{1}{24}\xi_1^4\xi_3 + \tfrac{1}{720}\xi_1^6\xi_2 + \tfrac{1}{362\,880}\xi_1^9)$$

$$q_{10} = \xi_1\xi_5 + \xi_2\xi_4 + \tfrac{1}{2}\xi_3^2 + \tfrac{1}{2}\xi_1^2\xi_2\xi_3 + \tfrac{1}{6}\xi_1\xi_2^3 + \tfrac{1}{6}\xi_1^3\xi_4 + \tfrac{1}{48}\xi_1^4\xi_2^2 + \tfrac{1}{120}\xi_1^5\xi_3$$

$$+ \tfrac{1}{5040}\xi_1^7\xi_2 + \tfrac{1}{3628\,800}\xi_1^{10}. \tag{A.2}$$

The exponential integral (Abramowitz and Stegun 1965) may be computed either from its power series

$$E_1(X) = -\ln X - \gamma - \sum_{n=1}^{\infty} \frac{(-X)^n}{(n!)n} \tag{A.3}$$

where γ is Euler's constant, or from the continued fraction

$$E_1(X) = \frac{e^{-X}}{X+} \frac{1}{1+} \frac{1}{X+} \frac{2}{1+} \frac{2}{X+} \cdots \frac{k}{1+} \frac{k}{X+} \cdots \tag{A.4}$$

which is valid for $|\arg X| < \pi$. (This condition is satisfied, since $\operatorname{Re} X = \tilde{M}^{1/2} \operatorname{Re} \xi = \tilde{M}^{1/2} \varepsilon > 0$.) By numerical experimentation with both representations, we found that a uniform accuracy (relative error less than 1%) may be achieved for $\operatorname{Re} X \geqslant 0$ by using (A.3) truncated at $n = 5$ when $|X| < 0.4$, and (A.4) truncated at $k = 30$ when $|X| > 0.4$.

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